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APPLICATION OF THE PONTRYAGIN MAXIMUM PRINCIPLE TO FLIGHT IN A VACUUM

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This paper discusses methods of determining fuel-optimum ascent trajectories in a vacuum (classical calculus of variations, Pontryagin maximum principle, dynamic programming, gradient method). It is found that the Pontryagin maximum principle has distinct advantages over the other methods. The mathematical relationships for the application of the principle to the title problem are derived. The difficulties encountered are discussed.

1. Notation /9*

g	[m sec^{-2}]	Acceleration of gravity
h	[m]	Altitude of flight
m	[kp/m sec^2]	Mass of the rocket
$\dot{m}_{max}$	[kp/m sec]	Maximum fuel mass flow
r_e	[m]	Radius of the Earth
S, S_{max}	[kp]	Propulsion unit thrust, propulsion unit maximum thrust
t	[s]	Time
V	[m/sec]	Flight losses
V_r	[m/sec]	Exhaust velocity of the fuel
α	[°]	Angle of inclination of the propulsion unit axis with respect to the trajectory
γ	[°]	Angle between the trajectory and the horizon
δ	[1]	Steering function of the magnitude of propulsion unit thrust
ψ		Adjoint variable

2. Statement of the Problem /10

The investigation was carried out within the framework of a determination

* Numbers in the margin indicate pagination in the original foreign text.

of optimum ascent trajectories for a two-stage space rocket. The problem was to place the second stage of the rocket in a circular orbit around the earth in such a way that the required fuel consumption is at a minimum.

In order to obtain approximate values for our project in as short a time as possible, we chose the following approach:

1. Experimental determination of the optimum ascent trajectory within the atmosphere (with the aid of an analog computer investigation).

2. Optimization of the trajectory in a vacuum according to principles of variational calculus (also with the aid of an analog computer investigation).

In this investigation we made the assumption that the staging separation occurs at altitudes at which the only external force is the earth's attraction.

The optimization of the overall trajectory for given initial and terminal conditions (at $h = 0$; at $h = h_{circular\ orbit}$) can be carried relatively easily using the division of the problem into two partial problems as mentioned above. The end points of the lower trajectory segment (staging position) as well as the initial values of the upper trajectory segment (staging position) are both considered as variable. The determination of the optimum overall trajectory, including the determination of the optimum staging position, then occurs by connection of the two trajectory segments in a way that will minimize the fuel consumption.

The following considerations will only be concerned with the second trajectory segment, i.e., the optimization of the fuel consumption in a vacuum. /11

The following simplifying assumptions are made for this purpose:

a) The trajectories are two-dimensional.

b) The target orbit is a circular orbit.

c) Limitations for the restriction of the available phase space are not taken into account.

The following steering variables are used:

1. Propulsion unit thrust S; where $0 \leqq S \leqq S_{max}$ is satisfied.
2. Thrust direction α; α is arbitrary.

For the steering variables we made the simplifying assumption that their change per time unit is not bounded.

3. Discussion of the Possible Methods of Solution

The applicable mathematical optimization procedure should satisfy the following conditions:

1. The optimization procedure must be such that by using a procedure which is in principle the same it becomes possible to solve more extensive problems of the same type. There is validity in this condition, because it enables us to accumulate information on this problem.

2. The method should have relatively small demands in time and cost (for the present stage of the investigation).

Due to the accuracy requirements for the results, it seemed appropriate to carry out the investigation using an analog computer. On the other hand, the use of the analog computer as a calculation aid satified the conditions set forth in 3.2. . /12

In principle, the following methods of solution are possible:

a) Classical variational calculus
b) Pontryagin maximum principle
c) Dynamic programming (Bellman)
d) Gradient method.

It should be noted that the most important difficulties in the application of the methods of Group a and Group b occur due to the boundary value problem which must be solved. On the other hand, difficulties occur in the application of methods c and d as the number of variables is increased (dimension of the phase space, number of steering variables).

Since, according to the requirement 3.1., it is required that, in principle, the same method is to be used for more extensive and more complicated problems (trajectory optimization within the atmosphere), we decided to avoid the use of methods c and d for the reasons given above. The methods a and b have the additional advantage that in the present problem the associated major difficulty, i.e., the solution of boundary value problems, disappears at least in part due to the variable staging position which has been assumed at least temporarily. Of the two optimization procedures which remain, the Pontryagin maximum principle was used for the solution of the problem. This procedure has the following advantages over the classical calculus of variations:

1. It expresses the content of three necessary conditions of the classical calculus of variations -- Euler differential equation, Legendre condition, Weierstrass necessary condition -- in a simple way, which is the necessary condition for the presence of an optimum. /13

2. In the present problem, there are no additional conditions which must be satisfied (classical calculus of variations: Weierstrass-Erdmann-corner conditions) due to the presence of discontinuities in the time

derivatives of the phase variables.

3. For more complicated problems (restriction of the allowable phase space) the difficulties in the application of classical calculus of variations increase in a manner which is disproportionally larger.

Since, in the present case we intended to attempt the solution of more complicated and more extensive problems, it seemed appropriate to prefer the maximum principle according to condition 3.1. .

4. Solution of the Optimization Problem

The quantity $x_0 = \frac{1}{m}$ was optimized in such a way that in circular orbit $\frac{1}{m}$ becomes a minimum.

The following are phase space variables: v, γ, h, m.

4.1. Formulation of the system of equations

The steering variables must be chosen in such a way that the functional

$$X_0 = \left(\frac{1}{m}\right) = \int_{t_0}^{t_1} \frac{|\dot{m}_{max}|}{m^2}\,\delta \cdot dt = \int_{t_0}^{t_1} f_{x_0} \cdot dt$$

becomes a minimum.

The following side conditions must be satisfied: /14

$$\frac{dv}{dt} = f_v = \frac{|\dot{m}_{max}|}{m} \cdot \delta \cdot v_T \cdot \cos\alpha - g \cdot \sin\gamma$$

$$\frac{d\gamma}{dt} = f_\gamma = \frac{|\dot{m}_{max}|}{m} \cdot \delta \cdot v_T \cdot \sin\alpha - \frac{g}{V} \cdot \cos\gamma + \frac{V}{r} \cdot \cos\gamma$$

$$\frac{dm}{dt} = f_m = -|\dot{m}_{max}| \cdot \delta$$

$$\frac{dh}{dt} = f_h = V \cdot \sin\gamma$$

The acceleration of gravity is calculated from

$$g = \frac{9{,}81 \cdot r_E^2}{(r_E + h)^2}$$

The system of differential equations for the adjoint variables results in (see Ref. 1):

$$\frac{d\psi_v}{dt} = -\psi_\gamma\left[-\frac{|\dot{m}_{max}|}{m \cdot v^2} \cdot \delta \cdot v_T \cdot \sin\alpha + \cos\gamma\left(\frac{g}{v^2} + \frac{1}{r}\right)\right] - \psi_h \cdot \sin\gamma$$

$$\frac{d\psi_\gamma}{dt} = \psi_v \cdot g \cdot \cos\gamma - \psi_\gamma \cdot \sin\gamma \cdot \left(\frac{g}{v} - \frac{v}{r}\right) - \psi_h \cdot v \cdot \cos\gamma$$

$$\frac{d\psi_h}{dt} = -\psi_v \cdot \frac{2 \cdot g}{r} \cdot \sin\gamma - \psi_\gamma \cos\gamma \left(\frac{2 \cdot g}{r\,v} - \frac{v}{r^2}\right)$$

$$\frac{d\psi_m}{dt} = \psi_v \frac{|\dot{m}_{max}|}{m} \delta \cdot v_T \cdot \cos\alpha + \psi_\gamma \frac{|\dot{m}_{max}|}{m^2 \cdot v} \cdot \delta v_T \cdot \sin\alpha + \psi_o \frac{2\,|\dot{m}_{max}|}{m^3} \cdot \delta$$

$$\frac{d\psi_{x_0}}{dt} = 0$$

For convenience (section 4.3.) a time transformation was carried out:

$$\tau = t_1 - t \quad ; \qquad d\tau = -dt$$

4.2. <u>Realization of the optimization criterion based on the maximum principle</u>. /15

In order that the steering variables α, δ and the phase variables v, γ, m, n are optimum in the sense that x_0 is made a minimum by them, it is necessary that the following conditions are satisfied (see Ref. 1) and that the vector function $\vec{\psi}$ (ψ_0, ψ_v, ψ_γ, ψ_m, ψ_n) does not vanish:

a) For each t, $t_0 \leq t \leq t_1$ the function

$$H(\psi\ ; v, \gamma, m, h\ ; \alpha, \delta) = \psi_{x_0} \cdot f_{x_0} + \psi_v \cdot f_v + \psi_\gamma \cdot f_\gamma + \psi_m \cdot f_m + \psi_h \cdot f_h$$

of the variables α, δ ($0 \leq \delta \leq 1$) takes on its maximum at the point $\alpha = \alpha(t)$, $\delta = \delta(t)$:

$$H = M(\psi\ ; v, \gamma, m, h)$$

$$M = \sup_{\alpha, 0 \leq \delta \leq 1} H(\psi\ ; v, \gamma, m, h\ ; \alpha, \delta)$$

b) At the time t_1 the following conditions must be satisfied:

$$\psi_{x_0}(t_1) \leq 0 \qquad\qquad M = 0$$

c) The differential equations given in section 4.1. must be satisfied. /16

In the present problem we have:

$$H = \psi_{x_0} \cdot \frac{|\dot{m}_{max}|}{m^2} \delta + \psi_v \frac{|\dot{m}_{max}|}{m} \cdot \delta \cdot v_T \cos\alpha - \psi_v \cdot g \cdot \sin\gamma$$

$$+ \psi_\gamma \frac{|\dot{m}_{max}|}{m \cdot v} \delta \cdot v_T \sin\alpha + \psi_\gamma \left(\frac{v}{r} - \frac{g}{v}\right) \cos\gamma - \psi_m |\dot{m}_{max}| \delta$$

$$+ \psi_h\, v \cdot \sin\gamma$$

after introduction of the switching function

$$A = \psi_{x_0} \cdot \frac{|\dot{m}_{max}|}{m^2} - \psi_m |\dot{m}_{max}| + \frac{|\dot{m}_{max}|}{m} \cdot v_T \left(\psi_v \cdot \cos\alpha + \frac{\psi_\gamma}{V} \cdot \sin\alpha\right)$$

the following result is obtained:

$$H = \delta \cdot A - \psi_V \cdot g \cdot \sin\gamma + \psi_\gamma \left(\frac{V}{r} - \frac{g}{V} \right) \cos\gamma + \psi_h \cdot V \cdot \sin\gamma$$

The criterion for the switching on and switching off, respectively, of the propulsion system follows from condition (a) of the maximum principle:

$$A > 0 \longrightarrow \delta = \delta_{max} = 1$$

$$A < 0 \longrightarrow \delta = \delta_{min} = 0$$

The thrust direction (α) follows from condition (a) as well:

$$\frac{\partial H}{\partial \alpha} = \frac{\partial A}{\partial \alpha} = \left[-\psi_V \cdot \sin\alpha + \frac{\psi_\gamma}{V} \cos\alpha \right] \cdot \frac{|\dot{m}_{max}|}{m} \cdot v_T = 0$$

$$\alpha = \arctan \frac{\psi_\gamma}{V \cdot \psi_V}$$

The necessary condition for satisfying the condition (a) with respect to α is the following: /17

$$\frac{\partial^2 H}{\partial \alpha^2} = \frac{\partial^2 A}{\partial \alpha^2} = \left[-\psi_V \cdot \cos\alpha - \frac{\psi_\gamma}{V} \sin\alpha \right] \cdot \frac{|\dot{m}_{max}|}{m} \cdot v_T$$

$$= -\psi_V \cdot \cos\alpha \left(1 + \frac{\psi_\gamma}{V \cdot \psi_V} \cdot \tan\alpha \right) \cdot \frac{|\dot{m}_{max}|}{m} \cdot v_T$$

$$= -\psi_V \cos\alpha \left[1 + \left(\frac{\psi_\gamma}{V \cdot \psi_V} \right)^2 \right] \cdot \frac{|\dot{m}_{max}|}{m} \cdot v_T < 0$$

Under the condition that $|\alpha| < \frac{\Pi}{2}$ it follows from this that:

$$\psi_V > 0$$

Summarizing, the following conditions result for optimum flight trajectories:

1. The system of differential equations of section 4.1. is satisfied.

2. Switching function for the thrust

$$A > 0 \longrightarrow \delta = 1$$

$$A < 0 \longrightarrow \delta = 0$$

3. Thrust direction $\alpha = \arctan \frac{\psi_\gamma}{V \cdot \psi_V}$, $\psi_V > 0$

4. $\psi_{x_0} = -1$ (the magnitude of ψ_{x0} can be chosen arbitrarily, because the system of differential equations is homogenous in ψ_k ($K = V, \gamma, m, n, x_0$).

5. $M|_{t1} = 0$.

4.3 Discussion of special features. /18

An analog computer was used as the computation aid for the reasons given in the introduction. The analog computer is particularly well suited for the solution of the system of equations, especially because the steering variables can be expressed in an explicit manner. The introduction of the functional in the form given has the advantage over the explicit condition $m_{\text{circular orbit}} = m_{\text{maximum circular orbit}}$, which seems simpler at first glance, in that the corresponding adjoint variable ψ_{x_0} is independent of the time. Thus, every calculated trajectory satisfies the necessary optimum condition and the course of ψ_{x0} need not be controlled.

We found that the switching function of the thrust, A, had to be simulated in a relatively exact manner. Deviations from the exact value, which are usually caused by non-linear calculation elements, result in high frequency on-off switching commands for the propulsion unit thrust at the time of switching.

The reversal of the direction of integration (time transformation) resulted from the fact that the final conditions (circular orbit) were given in the form of a point in phase space, whereas, in contrast to this, the initial conditions (staging position) must only lie within a region of phase space. Due to these facts it was possible to avoid extensive recursion formulas for the calculation of fixed end values by making use of the time transformation. In order to assure that the trajectory ends in the region of possible staging positions, the initial values (circular orbit) of three adjoint variables were varied manually (two of the five initial values of /19
the adjoint variables follow from the necessary condition already:

$\psi_{x0} = -1$; an additional initial value follows from $M|_{t1} = 0$).

A further difficulty resulted from the fact that when the circular orbit condition was adhered to ($\gamma = 0$, $h = h_{\text{circular orbit}}$, $V = V_{\text{circular orbit}}$) the condition $A = 0$ followed at the initial point of the integration interval. This corresponds to the trivial solution which states that the initial circular orbit is an optimum trajectory with respect to fuel consumption. Thus, it was necessary to deviate from the exact circular orbit condition as the initial state: the initial velocity was taken as $V_o = 0.995 \times V_{\text{circular orbit}}$; in addition, the free initial conditions of the adjoint variables were chosen in such a way that $A_{(t = t_1)} > 0$. The field of the optimum trajectories was not changed because, when the trajectory intercepts the circular orbit, at least a short time is required for the propulsion unit to be turned on.

5. Summary

This article showed a possible method of attack for the determination of fuel optimum trajectories in a vacuum, which can be carried out with a tolerable amount of effort. Such investigations are most often carried out during the project stage of a mission, where it is necessary to make a decision for one of the possible solutions.

The possible methods of optimization were discussed with regard to their applicability to the special problem of determining fuel optimum ascent trajectories. It was found that the Pontryagin maximum principle offers advantages over other methods. Then the mathematical relationships were derived based on the maximum principle as the optimization method. Finally, /20
we reported on peculiarities which occur when the investigations are carried out.

REFERENCES

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